Int. J. Solids Structures Vol. 27, No. 3, pp. 343–354, 1991
Printed in Great Britain.

0020–7683/91 $3.00 + .00
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EXTERNAL CIRCULAR CRACK UNDER CONCENTRATED ANTISYMMETRIC LOADING

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(*Received* 13 *October* 1989; *in revised form* 10 *January* 1990)

Abstract—A complete solution is given for the first time to the title problem. Explicit expressions are derived for the field of stresses and displacement in a transversely isotropic space weakened by an external circular crack and subjected to two antisymmetrically applied concentrated forces. The method is based on the new results in potential theory obtained by the author earlier. The presented results may be used as Green's functions for a general case of antisymmetric loading so that the complete solution can be presented in quadratures.

INTRODUCTION

The external circular crack may be perceived as two elastic half-spaces connected in the plane $z = 0$ by a circular domain which is called hereafter the crack neck. Ufliand (1967) was, probably, the first to consider the equilibrium of an *isotropic* elastic body weakened by an external circular crack and subjected to the action of two antisymmetric normal forces by an integral transform method. The same problem for the case of *transversely isotropic* body was solved in Fabrikant (1971). All these solutions define the elastic field in the plane $z = 0$ only. We call a solution *complete* when the explicit expressions are given for the stresses and displacements all over the elastic space. One may argue that since the stresses exerted in the crack neck are known, we can substitute them into the Boussinesq point force solution (which is well known, for example, see Fabrikant, 1970) and obtain the complete solution in quadratures. Theoretically, this can be done, but practically, this solution would be of little use since it would require double integration, with the integrand being singular. The computing time for this procedure would be quite significant, and its accuracy would be very doubtful. This is the main reason why, to the best of my knowledge, nobody has tried so far to obtain a complete solution, even in the case of an isotropic body. On the other hand, knowledge of the complete solution is of great interest since it is essential for consideration of more complicated problems. For example, using linear superposition of the solutions for symmetric and antisymmetric loading, we can obtain the solution to the problem of one-sided loading of a crack.

The complete solution has become possible due to the new results in potential theory obtained by the author, Fabrikant (1989). The expressions for the stresses in the crack neck are fed in the point force solution, with one important distinction: the integrals are computed in elementary functions and lead to remarkably simple and elementary expressions.

THEORY

Consider a transversely isotropic elastic body which is characterized by five elastic constants A_{ik} defining the following stress–strain relationships:

$$\sigma_x = A_{11}\frac{\partial u_x}{\partial x}+(A_{11}-2A_{66})\frac{\partial u_y}{\partial y}+A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_y = (A_{11}-2A_{66})\frac{\partial u_x}{\partial x}+A_{11}\frac{\partial u_y}{\partial y}+A_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z = A_{13}\frac{\partial u_x}{\partial x}+A_{13}\frac{\partial u_y}{\partial y}+A_{33}\frac{\partial w}{\partial z},$$

$$\tau_{xy} = A_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \tau_{yz} = A_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \tau_{zx} = A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u_x}{\partial z}\right). \tag{1}$$

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \tag{2}$$

Substitution of (1) in (2) yields:

$$A_{11}\frac{\partial^2 u_x}{\partial x^2} + A_{66}\frac{\partial^2 u_x}{\partial y^2} + A_{44}\frac{\partial^2 u_x}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_y}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0,$$

$$A_{66}\frac{\partial^2 u_y}{\partial x^2} + A_{11}\frac{\partial^2 u_y}{\partial y^2} + A_{44}\frac{\partial^2 u_y}{\partial z^2} + (A_{11} - A_{66})\frac{\partial^2 u_x}{\partial x \partial y} + (A_{13} + A_{44})\frac{\partial^2 w}{\partial y \partial z} = 0,$$

$$A_{44}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{44} + A_{13})\left[\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z}\right] = 0. \tag{3}$$

Introducing complex tangential displacements $u = u_x + \mathrm{i}u_y$, and $\bar{u} = u_x - \mathrm{i}u_y$ will allow us to reduce the number of equations in (3) by one, and to rewrite these equations in a more compact manner, namely,

$$\tfrac{1}{2}(A_{11} + A_{66})\Delta u + A_{44}\frac{\partial^2 u}{\partial z^2} + \tfrac{1}{2}(A_{11} - A_{66})\Lambda^2 \bar{u} + (A_{13} + A_{44})\Lambda\frac{\partial w}{\partial z} = 0,$$

$$A_{44}\Delta w + A_{33}\frac{\partial^2 w}{\partial z^2} + \tfrac{1}{2}(A_{13} + A_{44})\frac{\partial}{\partial z}(\bar{\Lambda} u + \Lambda \bar{u}) = 0. \tag{4}$$

Here the following differential operators were used:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Lambda = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}, \tag{5}$$

and the overbar indicates the complex conjugate value. Note also that $\Delta = \Lambda\bar{\Lambda}$. One can verify that eqns (4) can be satisfied by

$$u = \Lambda(F_1 + F_2 + \mathrm{i}F_3), \quad w = m_1\frac{\partial F_1}{\partial z} + m_2\frac{\partial F_2}{\partial z} \tag{6}$$

where all three functions F_k satisfy the eqn (Elliott, 1948):

$$\Delta F_k + \gamma_k^2\frac{\partial^2 F_k}{\partial z^2} = 0, \quad \text{for } k = 1, 2, 3, \tag{7}$$

and the values of m_k and γ_k are related by the following expressions (Elliott, 1948):

$$\frac{A_{44} + m_k(A_{13} + A_{44})}{A_{11}} = \frac{m_k A_{33}}{m_k A_{44} + A_{13} + A_{44}} = \gamma_k^2, \quad \text{for } k = 1, 2; \quad \gamma_3 = (A_{44}/A_{66})^{1/2}. \tag{8}$$

Introducing the notation $z_k = z/\gamma_k$, for $k = 1, 2, 3$, we may call function $F_k = F(x, y, z_k)$ harmonic. Note the property $m_1 m_2 = 1$, which seems to have escaped the attention of previous researchers, and which will help us to simplify various expressions to follow. The other elastic constants which will be used throughout the paper are:

$$G_1 = \beta+\gamma_1\gamma_2 H, \quad G_2 = \beta-\gamma_1\gamma_2 H,$$

$$H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)}, \quad \alpha = \frac{(A_{11}A_{33})^{1/2}-A_{13}}{A_{11}(\gamma_1+\gamma_2)}, \quad \beta = \frac{\gamma_3}{2\pi A_{44}}. \tag{9}$$

Introduce the following inplane stress components:

$$\sigma_1 = \sigma_x+\sigma_y, \quad \sigma_2 = \sigma_x-\sigma_y+2i\tau_{xy}, \quad \tau_z = \tau_{zx}+i\tau_{yz}. \tag{10}$$

This will simplify expressions (1), namely

$$\sigma_1 = (A_{11}-A_{66})(\bar{\Lambda}u+\Lambda\bar{u})+2A_{13}\frac{\partial w}{\partial z}, \quad \sigma_2 = 2A_{66}\Lambda u,$$

$$\sigma_z = \tfrac{1}{2}A_{13}(\bar{\Lambda}u+\Lambda\bar{u})+A_{33}\frac{\partial w}{\partial z}, \quad \tau_z = A_{44}\left[\frac{\partial u}{\partial z}+\Lambda w\right]. \tag{11}$$

We have now only four components of stress, instead of six, as it was in (1). The substitution of (6) in (11) yields:

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2-(1+m_1)\gamma_3^2]F_1+[\gamma_2^2-(1+m_2)\gamma_3^2]F_2\},$$

$$\sigma_2 = 2A_{66}\Lambda^2(F_1+F_2+iF_3),$$

$$\sigma_z = A_{44}\frac{\partial^2}{\partial z^2}[(1+m_1)\gamma_1^2F_1+(1+m_2)\gamma_2^2F_2] = -A_{44}\Delta[(1+m_1)F_1+(1+m_2)F_2],$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1+(1+m_2)F_2+iF_3]. \tag{12}$$

Here we used the fact that each F_k satisfies eqn (7), and the relation: $A_{11}\gamma_k^2-A_{13}m_k = A_{44}(1+m_k)$, (for $k = 1, 2$) which is an immediate consequence of (8). Expressions (6) and (12) give a general solution, expressed in terms of three harmonic functions F_k. It is very attractive to express each function F_k through just *one* harmonic function as follows:

$$F_k(x, y, z) = c_k F(x, y, z_k),$$

where $z_k = z/\gamma_k$, and c_k is an as yet unknown complex constant. As we shall see further, this is indeed possible. All the results obtained in the paper are valid for isotropic solids, provided that we take

$$\gamma_1 = \gamma_2 = \gamma_3 = 1, \quad H = \frac{1-\nu^2}{\pi E}, \quad \alpha = \frac{1-2\nu}{2(1-\nu)},$$

$$\beta = \frac{1+\nu}{\pi E}, \quad G_1 = \frac{(2-\nu)(1+\nu)}{\pi E}, \quad G_2 = \frac{\nu(1+\nu)}{\pi E}, \tag{13}$$

where E is the elastic modulus, and ν is Poisson's coefficient.

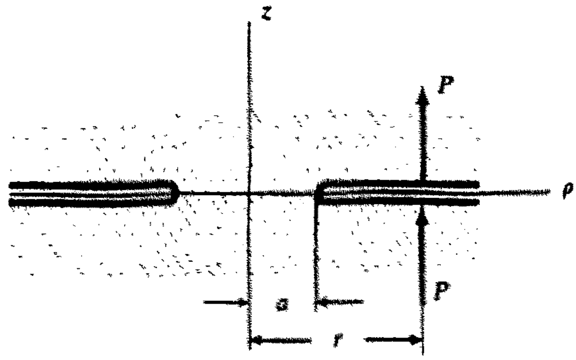

Fig. 1. The crack geometry.

We consider a transversely isotropic elastic space weakened by an external circular crack of radius a in the plane $z = 0$ (Fig. 1). Let two point forces P be applied to the crack faces antisymmetrically in the Oz direction at the points with cylindrical coordinates $(r, \psi, 0^+)$ and $(r, \psi, 0^-)$. The problem, due to antisymmetric loading, can be reduced to that of a half-space $z \geqslant 0$, with the boundary conditions at the plane $z = 0$

$$\begin{aligned}
&u = 0, && \text{for } 0 \leqslant \rho \leqslant a, && 0 \leqslant \phi < 2\pi;\\
&\sigma = 0, && \text{for } 0 \leqslant \rho \leqslant a, && 0 \leqslant \phi < 2\pi;\\
&\sigma = P\delta(\rho - r, \phi - \psi)/\rho, && \text{for } a \leqslant \rho \leqslant \infty, && 0 \leqslant \phi < 2\pi;\\
&\tau = 0, && \text{for } a \leqslant \rho \leqslant \infty, && 0 \leqslant \phi < 2\pi.
\end{aligned} \tag{14}$$

Here $\sigma = -\sigma_z$ and $\tau = -\tau_z$ as they are defined in (12). It is known (Fabrikant, 1989) that in the case of a transversely isotropic elastic half-space subjected to a general concentrated force with the components T_x, T_y and P, the complete solution can be expressed through the three potential functions:

$$F_1 = \frac{H\gamma_1}{m_1 - 1}[\tfrac{1}{2}\gamma_2(\bar{\Lambda}\chi_1 + \Lambda\bar{\chi}_1) + P\ln(R_1 + z_1)],$$

$$F_2 = \frac{H\gamma_2}{m_2 - 1}[\tfrac{1}{2}\gamma_1(\bar{\Lambda}\chi_2 + \Lambda\bar{\chi}_2) + P\ln(R_2 + z_2)],$$

$$F_3 = \mathrm{i}\frac{\gamma_3}{4\pi A_{44}}(\bar{\Lambda}\chi_3 - \Lambda\bar{\chi}_3). \tag{15}$$

Here (ρ_0, ϕ_0) is the point of the boundary where the concentrated force is applied;

$$\chi_k(z) = \chi(z_k), \quad R_k = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z_k^2]^{1/2}, \quad \text{for } k = 1, 2, 3;$$
$$\chi(z) = T[z\ln(R_0 + z) - R_0], \quad T = T_x + \mathrm{i}T_y, \quad R_0 = [\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0) + z^2]^{1/2}. \tag{16}$$

Substitution of (15)–(16) in (6) yields

$$u = \frac{\gamma_3}{4\pi A_{44}}\left[\frac{T}{R_3} + \frac{q^2\bar{T}}{R_3(R_3 + z_3)^2}\right] + \frac{H\gamma_2}{m_2 - 1}\left\{\frac{1}{2}\gamma_1\left[-\frac{T}{R_2} + \frac{q^2\bar{T}}{R_2(R_2 + z_2)^2}\right] + \frac{Pq}{R_2(R_2 + z_2)}\right\}$$
$$+ \frac{H\gamma_1}{m_1 - 1}\left\{\frac{1}{2}\gamma_2\left[-\frac{T}{R_1} + \frac{q^2\bar{T}}{R_1(R_1 + z_1)^2}\right] + \frac{Pq}{R_1(R_1 + z_1)}\right\}, \tag{17}$$

$$w = H\left\{\frac{1}{2}(T\bar{q}+\bar{T}q)\left[\frac{\gamma_2 m_1}{(m_1-1)R_1(R_1+z_1)}+\frac{\gamma_1 m_2}{(m_2-1)R_2(R_2+z_2)}\right]\right.$$
$$\left.+P\left[\frac{m_1}{(m_1-1)R_1}+\frac{m_2}{(m_2-1)R_2}\right]\right\}. \quad (18)$$

Here

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}. \quad (19)$$

Expressions (17) and (18) simplify for the case when $z = 0$

$$u = \tfrac{1}{2}G_1\frac{T}{R}+\tfrac{1}{2}G_2\frac{\bar{T}q^2}{R^3}-H\alpha\frac{P}{\bar{q}}, \quad (20)$$

$$w = H\alpha\mathscr{R}\left(\frac{T}{q}\right)+H\frac{P}{R}. \quad (21)$$

Here $\mathscr{R}$ is the real part sign; H, α, G_1, and G_2 are defined by (9), and

$$R = [\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}. \quad (22)$$

Expression (20) can be used for the integral equation formulation of the problem. The governing integral equation will take the form (Fabrikant, 1971)

$$\frac{G_1}{2}\int_0^{2\pi}\int_0^a\frac{\tau(\rho_0,\phi_0)}{R}\rho_0\,d\rho_0\,d\phi_0+\frac{G_2}{2}\int_0^{2\pi}\int_0^a\frac{q^2\bar{\tau}(\rho_0,\phi_0)}{R^3}\rho_0\,d\rho_0\,d\phi_0 = \frac{H\alpha P}{\rho e^{-i\phi}-re^{-i\psi}}. \quad (23)$$

Here τ stands for τ_z as it was defined in (10). This equation has been solved in Fabrikant (1971), and its exact solution reads

$$\tau(\rho,\phi) = -\frac{2PH\alpha}{\pi^2 G_1 re^{-i\psi}(a^2-\rho^2)^{1/2}}\frac{1}{1-\zeta}\left[1+\left(\frac{\zeta}{1-\zeta}\right)^{1/2}\sin^{-1}\sqrt{\zeta}\right]. \quad (24)$$

Here $\zeta = \rho e^{-i\phi}/(re^{-i\psi})$. Expressions (15) can be used to obtain formulae for the potential functions in the case of a distributed loading. This will lead to computation of various integrals involving (24) and some functions of distance between points (see, for example, (17) and (18)). The simplest integral to compute is

$$I = \int_0^{2\pi}\int_0^a\frac{\tau(\rho_0,\phi_0)}{R_0}\rho_0\,d\rho_0\,d\phi_0. \quad (25)$$

Here τ is defined by (24), and R_0 is given by (16). Let us make use of the integral representation (Fabrikant, 1989)

$$\frac{1}{R_0} = \frac{2}{\pi}\int_0^{\rho_0}\lambda\left(\frac{l_1^2(x)}{\rho\rho_0},\phi-\phi_0\right)\frac{[l_2^2(x)-x^2]^{1/2}\,dx}{(r^2-x^2)^{1/2}[l_2^2(x)-l_1^2(x)]},$$

$$l_1(x) = \tfrac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}-[(\rho-x)^2+z^2]^{1/2}\},$$

$$l_2(x) = \tfrac{1}{2}\{[(\rho+x)^2+z^2]^{1/2}+[(\rho-x)^2+z^2]^{1/2}\},$$

$$\lambda(k,\psi) = \frac{1-k^2}{1+k^2-2k\cos\psi}, \tag{26}$$

and the series expansion for (24), namely,

$$\tau(\rho_0,\phi_0) = -\frac{2PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\frac{(\rho_0 e^{-i\phi_0})^n}{(re^{-i\psi})^{n+1}(a^2-\rho_0^2)^{1/2}}. \tag{27}$$

Substitution of (26) and (27) in (25) yields, after integration with respect to ϕ_0

$$I = -\frac{8PH\alpha}{\pi^{3/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)(re^{-i\psi})^{n+1}}\int_0^a \rho_0\,\mathrm{d}\rho_0\int_0^{\rho_0}\frac{[l_2^2(x)-x^2]^{1/2}(l_1^2(x)e^{-i\phi}/\rho)^n\,\mathrm{d}x}{(\rho_0^2-x^2)^{1/2}(a^2-\rho_0^2)^{1/2}[l_2^2(x)-l_1^2(x)]}.$$

Changing the order of integration and consequent integration with respect to ρ_0 gives

$$I = -\frac{4PH\alpha}{\pi^{1/2}G_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)(re^{-i\psi})^{n+1}}\int_0^a\frac{[l_2^2(x)-x^2]^{1/2}(l_1^2(x)e^{-i\phi}/\rho)^n\,\mathrm{d}x}{l_2^2(x)-l_1^2(x)}. \tag{28}$$

The summation in (28) can be performed, with the result

$$I = -\frac{4PH\alpha\rho e^{i\phi}}{\pi G_1}\int_0^a\left[1+\frac{l_1(x)}{[b^2-l_1^2(x)]^{1/2}}\sin^{-1}\left(\frac{l_1(x)}{b}\right)\right]\frac{[l_2^2(x)-x^2]^{1/2}\,\mathrm{d}x}{[l_2^2(x)-l_1^2(x)][b^2-l_1^2(x)]}. \tag{29}$$

By introducing a new variable $y = l_1(x)$, $x = y[1+z^2/(\rho^2-y^2)]^{1/2}$, the integral (29) will take the form

$$I = -\frac{4PH\alpha\rho e^{i\phi}}{\pi G_1}\int_0^{l_1}\left[1+\frac{y}{(b^2-y^2)^{1/2}}\sin^{-1}\left(\frac{y}{b}\right)\right]\frac{\mathrm{d}y}{(\rho^2-y^2)^{1/2}(b^2-y^2)}.$$

Throughout this paper the abbreviations l_1 and l_2 denote $l_1(a)$ and $l_2(a)$ respectively. The last integral can be computed in an elementary manner, and the final result is

$$I = \frac{4PH\alpha}{\pi G_1 q}\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right]. \tag{30}$$

Here q is defined by (19), and $b = \rho e^{i(\phi-\psi)}$. In order to find the main potential functions (15), we need to compute the integral

$$I_1 = \int_0^{2\pi}\int_0^a\frac{\bar{q}\tau(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0}{R_0+z}. \tag{31}$$

This integral can be computed from (30) by means of application of the operator $\bar{\Lambda}$ to both sides of (30) and consequent integration of the result twice with respect to z. Application of $\bar{\Lambda}$ to (30) yields the following integral

$$\int_0^{2\pi}\int_0^a\frac{\bar{q}}{R_0^3}\tau(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = -\frac{4PH\alpha}{\pi G_1}\frac{l_1(\rho^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)(b^2-l_1^2)}\left[1+\frac{l_1}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right]. \tag{32}$$

Integration of both sides of (32) with respect to z results in

$$\int_0^{2\pi}\int_0^a \frac{\bar{q}}{R_0(R_0+z)}\tau(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = -\frac{4PH\alpha}{\pi G_1}\left\{\frac{1}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]+\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\,\mathrm{d}x}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}\right\}. \quad (33)$$

Various formulae from the Appendix were used in the intermediary transformations. Yet another integration of (33) with respect to z gives

$$\int_0^{2\pi}\int_0^a \frac{\bar{q}}{R_0+z}\tau(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0 = \frac{4PH\alpha}{\pi G_1}\left\{\frac{z}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)\right.\right.$$

$$\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)$$

$$\left.+z\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\,\mathrm{d}x}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}-\int_0^{l_1}\frac{x\sin^{-1}(x/b)\,\mathrm{d}x}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}\right\}. \quad (34)$$

The last result allows us to define the potential functions (15) as follows:

$$F_1 = \frac{PH\gamma_1}{m_1-1}\left\{-\frac{2H\gamma_2\alpha}{\pi G_1}[f(z_1)+\bar{f}(z_1)]+\ln(R_1+z_1)\right\},$$

$$F_2 = \frac{PH\gamma_2}{m_2-1}\left\{-\frac{2H\gamma_1\alpha}{\pi G_1}[f(z_2)+\bar{f}(z_2)]+\ln(R_2+z_2)\right\},$$

$$F_3 = \frac{\mathrm{i}\gamma_3PH\alpha}{\pi^2A_{44}G_1}[f(z_3)-\bar{f}(z_3)]. \quad (35)$$

Here the notation was introduced

$R_k = [\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z_k^2]^{1/2}$, $z_k = z/\gamma_k$, for $k = 1, 2, 3$;

$$f(z) = \frac{z}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)$$

$$+z\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\,\mathrm{d}x}{(a^2-x^2)^{1/2}(b^2-x^2)^{3/2}}-\int_0^{l_1}\frac{x\sin^{-1}(x/b)\,\mathrm{d}x}{(\rho^2-x^2)^{1/2}(b^2-x^2)^{1/2}}. \quad (36)$$

Now the complete solution can be obtained by substitution of (35)–(36) into (6) and (12). The result is

$$u = PH\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\Lambda[f(z_k)+\bar{f}(z_k)]+\frac{q\gamma_k}{R_k(R_k+z_k)}\right\}$$

$$+\frac{\gamma_3PH\alpha}{\pi^2A_{44}G_1}\Lambda[f(z_3)-\bar{f}(z_3)], \quad (37)$$

$$w = PH\sum_{k=1}^{2}\frac{m_k}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1\gamma_k}\frac{\partial}{\partial z_k}[f(z_k)+\bar{f}(z_k)]+\frac{1}{R_k}\right\}. \quad (38)$$

$$\sigma_1 = 2PHA_{66}\sum_{k=1}^{2}\frac{1-(1+m_k)(\gamma_3/\gamma_k)^2}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\frac{\partial^2}{\partial z_k^2}[f(z_k)+\bar{f}(z_k)]-\frac{z}{R_k^3}\right\}. \quad (39)$$

$$\sigma_2 = 2PHA_{66}\sum_{k=1}^{2}\frac{1}{m_k-1}\left\{-\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\Lambda^2[f(z_k)+\bar{f}(z_k)]-\frac{\gamma_k q^2(2R_k+z_k)}{R_k^3(R_k+z_k)^2}\right\}$$
$$+\frac{2PH\alpha}{\pi^2 G_1\gamma_3}\Lambda^2[f(z_3)-\bar{f}(z_3)], \quad (40)$$

$$\sigma_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^k\left\{\frac{2H\alpha\gamma_1\gamma_2}{\pi G_1}\frac{\partial^2}{\partial z_k^2}[f(z_k)+\bar{f}(z_k)]+\frac{z}{R_k^3}\right\}, \quad (41)$$

$$\tau_z = \frac{P}{2\pi(\gamma_1-\gamma_2)}\sum_{k=1}^{2}(-1)^k\left\{\frac{2H\alpha\gamma_1\gamma_2}{\pi\gamma_k G_1}\Lambda\frac{\partial}{\partial z}[f(z_k)+\bar{f}(z_k)]+\frac{q}{R_k^3}\right\}$$
$$+\frac{PH\alpha}{\pi^2 G_1}\Lambda\frac{\partial}{\partial z_3}[f(z_3)-\bar{f}(z_3)]. \quad (42)$$

Here are the explicit expressions for various derivatives of f which will be needed

$$\frac{\partial f}{\partial z} = \frac{1}{(b^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(b^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(b^2-a^2)^{1/2}}\right)\right]$$
$$+\int_0^{l_1}\frac{x^2\sin^{-1}(x/b)\,dx}{(a^2-x^2)^{1/2}(b^2-x^2)^{1/2}}, \quad (43)$$

$$\Lambda f = \frac{1}{q}\left[\sin^{-1}\left(\frac{a}{l_2}\right)-\frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \quad (44)$$

$$\Lambda\bar{f} = \frac{l_1 e^{i\phi}(\rho^2-l_1^2)^{1/2}}{\rho(\bar{b}^2-l_1^2)} - \frac{z\rho_0 e^{i\phi_0}}{\bar{b}^2-a^2}\left\{\frac{1}{(\bar{b}^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2-a^2)^{1/2}}\right)\right.\right.$$
$$\left.\left.-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(\bar{b}^2-a^2)^{1/2}}\right)\right]+\frac{a}{\bar{b}^2}-\frac{(a^2-l_1^2)^{1/2}}{\bar{b}^2-l_1^2}\right\}+\frac{ze^{i\phi}}{\rho}\int_0^{l_1}\frac{x^4\sin^{-1}(x/\bar{b})\,dx}{[(a^2-x^2)(\bar{b}^2-x^2)]^{3/2}}, \quad (45)$$

$$\frac{\partial^2 f}{\partial z^2} = -\frac{l_1(\rho^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)(b^2-l_1^2)}\left[1+\frac{l_1}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \quad (46)$$

$$\frac{\partial}{\partial z}\Lambda f = \frac{\rho e^{i\phi}(a^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)(b^2-l_1^2)}\left[1+\frac{l_1}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right], \quad (47)$$

$$\frac{\partial}{\partial z}\Lambda\bar{f} = \frac{(a^2-l_1^2)^{1/2}}{\bar{b}^2-l_1^2}\left[\frac{\rho e^{i\phi}}{l_2^2-l_1^2}+\frac{\rho_0 e^{i\phi_0}}{\bar{b}^2-a^2}\right]-\frac{l_1^3 e^{i\phi}(\rho^2-l_1^2)\sin^{-1}(l_1/b)}{\rho(a^2-l_1^2)^{1/2}(\bar{b}^2-l_1^2)^{3/2}(l_2^2-l_1^2)}$$
$$-\frac{\rho_0 e^{i\phi_0}a}{\bar{b}^2(\bar{b}^2-a^2)}-\frac{\rho_0 e^{i\phi_0}}{(\bar{b}^2-a^2)^{3/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2-a^2)^{1/2}}\right)-\tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(\bar{b}^2-a^2)^{1/2}}\right)\right]$$
$$+\frac{e^{i\phi}}{\rho}\int_0^{l_1}\frac{x^4\sin^{-1}(x/b)\,dx}{[(a^2-x^2)(b^2-x^2)]^{3/2}}, \quad (48)$$

$$\Lambda^2 f = -\frac{2}{\bar{q}^2}\left[\sin^{-1}\left(\frac{a}{l_2}\right) - \frac{(\rho^2-l_1^2)^{1/2}}{(b^2-l_1^2)^{1/2}}\sin^{-1}\left(\frac{l_1}{b}\right)\right] - \frac{e^{i\phi}(\rho^2-l_1^2)^{1/2}}{\bar{q}(l_2^2-l_1^2)}\left\{\frac{l_1}{\rho} + \frac{\rho(a^2-l_1^2)}{l_1(b^2-l_1^2)} + \frac{\sin^{-1}\left(\frac{l_1}{b}\right)}{(b^2-l_1^2)^{1/2}}\left[\frac{l_2^2}{\rho} + \frac{\rho(a^2-l_1^2)}{b^2-l_1^2}\right]\right\}, \quad (49)$$

$$\Lambda^2 \bar{f} = \frac{z\rho_0^2 e^{2i\phi_0}}{(\bar{b}^2-a^2)^2}\left\{\frac{3}{(\bar{b}^2-a^2)^{1/2}}\left[\tan^{-1}\left(\frac{a}{(\bar{b}^2-a^2)^{1/2}}\right) - \tan^{-1}\left(\frac{(a^2-l_1^2)^{1/2}}{(\bar{b}^2-a^2)^{1/2}}\right)\right] + \frac{a(5\bar{b}^2-2a^2)}{\bar{b}^4} - \frac{(a^2-l_1^2)^{1/2}(5\bar{b}^2-2a^2-3l_1^2)}{(\bar{b}^2-l_1^2)^2}\right\} - \frac{2\rho_0 e^{i(\phi+\phi_0)}(\rho^2-l_1^2)^{1/2}}{(\bar{b}^2-l_1^2)^2}\left[\frac{l_1}{\rho} + \frac{\rho(a^2-l_1^2)}{l_1(l_2^2-l_1^2)}\right]$$
$$+ \frac{e^{2i\phi} l_1(\rho^2-l_1^2)^{1/2}}{(\bar{b}^2-l_1^2)(l_2^2-l_1^2)}\left[\frac{2l_1^2-\rho^2}{\rho^2} + \frac{2(a^2-l_1^2)}{\bar{b}^2-l_1^2} - \frac{l_1^3(\rho^2-l_1^2)\sin^{-1}(l_1/\bar{b})}{\rho^2(a^2-l_1^2)[\bar{b}^2-l_1^2]^{1/2}}\right] + \frac{3ze^{2i\phi}}{\rho^2}\int_0^{l_1}\frac{x^6\sin^{-1}(x/\bar{b})\,dx}{(a^2-x^2)^{5/2}(\bar{b}^2-x^2)^{3/2}}. \quad (50)$$

Formulae (37)–(50) represent the main new results of this paper. One can notice that some of the derivatives still contain uncomputed integrals, but the main advantage is that those integrals are single, rather than double, and that their integrands are non-singular which makes them easy to compute by any standard subroutine.

The main results are valid for isotropic bodies as well, provided that we substitute the elastic constants and compute the limits according to (13). These limits may be computed by using the L'Hôpital rule. The following scheme should be used:

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{f(z_1)}{m_1-1} + \frac{f(z_2)}{m_2-1}\right] = -f(z) - \frac{z}{2(1-\nu)}f'(z), \quad (51)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{m_1-1} + \frac{m_2 f(z_2)}{m_2-1}\right] = f(z) - \frac{z}{2(1-\nu)}f'(z), \quad (52)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{[1-(1+m_1)(\gamma_3/\gamma_1)^2]f(z_1)}{m_1-1} + \frac{[1-(1+m_2)(\gamma_3/\gamma_2)^2]f(z_2)}{m_2-1}\right] = \frac{2(1+\nu)f(z)+zf'(z)}{2(1-\nu)}, \quad (53)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\gamma_1 f(z_1)}{m_1-1} + \frac{\gamma_2 f(z_2)}{m_2-1}\right] = -\frac{(1-2\nu)f(z)+zf'(z)}{2(1-\nu)}, \quad (54)$$

$$\lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{m_1 f(z_1)}{\gamma_1(m_1-1)} + \frac{m_2 f(z_2)}{\gamma_2(m_2-1)}\right] = \frac{(1-2\nu)f(z)-zf'(z)}{2(1-\nu)}. \quad (55)$$

Here the following relationships were used

$$\lim_{\gamma_1\to\gamma_2\to 1} m_1 = 1, \quad \lim_{\gamma_1\to\gamma_2\to 1}\left[\frac{\partial m_1}{\partial \gamma_1}\right] = 2(1-\nu), \quad (56)$$

and the symbol (′) indicates differentiation with respect to z. The field of displacements in the case of isotropy will take the form

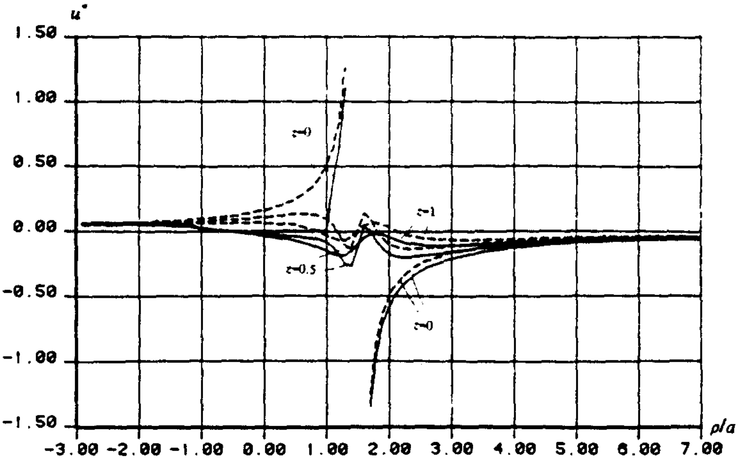

Fig. 2. The field of tangential displacements.

$$u = \frac{1+\nu}{2\pi E}P\left\{\frac{zq}{R_0^3} - \frac{(1-2\nu)q}{R_0(R_0+z)} + \frac{2}{\pi}(1-2\nu)\Lambda f(z) - \frac{2\nu(1-2\nu)}{\pi(2-\nu)}\Lambda\bar{f}(z) + \frac{1-2\nu}{\pi(2-\nu)}z\frac{\partial}{\partial z}\Lambda[f(z)+\bar{f}(z)]\right\}, \quad (57)$$

$$w = \frac{1+\nu}{2\pi E}P\left\{\frac{1-2\nu}{\pi(2-\nu)}[-(1-2\nu)[f'(z)+\bar{f}'(z)]+z[f''(z)+\bar{f}''(z)]] + \frac{2(1-2\nu)}{R_0} + \frac{z^2}{R_0^3}\right\}. \quad (58)$$

The derivation of the field of stresses for the case of isotropy is left to the reader.

It is of interest to investigate the influence of crack neck on the field of displacements. This can be done by comparison of (57)–(58) with the case of an elastic half-space subjected to a normal concentrated load P which is given by the last two terms in (57)–(58). As we can see, the most difference will be achieved in the case of Poisson coefficient $\nu = 0$, while in the other extreme, namely, $\nu = 1/2$, both solutions coincide. The computations were made for the case $\nu = 0$, $a = 2$, $r = 3$, $\psi = 0$, $\phi = 0$. The value of $u^* = (u/a)(2\pi E)/[P(1+\nu)]$ versus ρ/a for $z = 0$, $z = 0.5$ and $z = 1$ is given in Fig. 2. The negative value of ρ is understood as its value for $\phi = \pi$. A similar value of $w^* = (w/a)(2\pi E)/[P(1+\nu)]$ is presented in Fig. 3. In both figures, the solid line curves correspond to formulae (57) and (58)

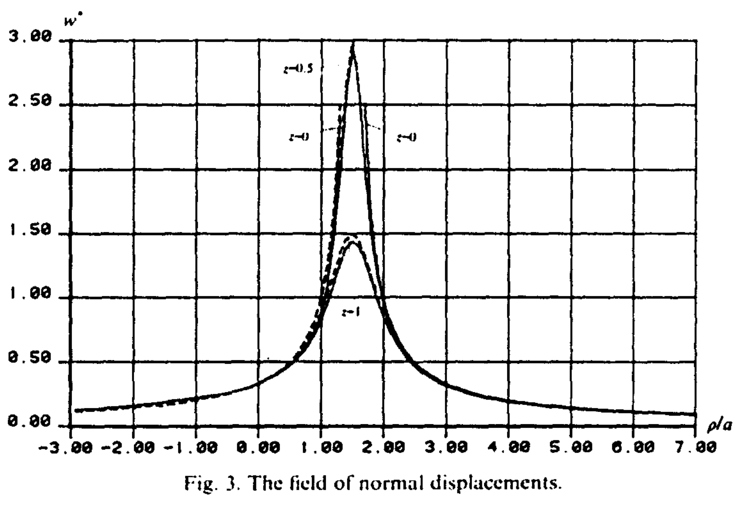

Fig. 3. The field of normal displacements.

respectively, while the dotted line curves describe the field in an elastic half-space subjected to a normal load only. As we can see, the field of normal displacements is practically unaffected even in this extreme case, while the field of tangential displacements differs significantly in the vicinity of the applied force and the crack neck. All the dotted curves in Fig. 2 go above the relevant solid line curves. A similar picture is observed in Fig. 3 for positive ρ, and it becomes reverse for ρ negative.

Acknowledgement—The reported research was sponsored by a grant from the Natural Sciences and Engineering Research Council of Canada.

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APPENDIX

We present here some mathematical results used in various transformations throughout this paper. The main properties of l_1 and l_2 are:

$$l_1 l_2 = a\rho, \quad l_1^2 + l_2^2 = a^2 + \rho^2 + z^2, \tag{A1}$$

$$(l_2^2 - \rho^2)^{1/2}(l_2^2 - a^2)^{1/2} = z l_2, \quad (a^2 - l_1^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z l_1,$$
$$(a^2 - l_1^2)^{1/2}(l_2^2 - a^2)^{1/2} = z a, \quad (l_2^2 - \rho^2)^{1/2}(\rho^2 - l_1^2)^{1/2} = z\rho. \tag{A2}$$

$$\frac{\partial l_1}{\partial z} = -\frac{z l_1}{l_2^2 - l_1^2}, \quad \frac{\partial l_2}{\partial z} = \frac{z l_2}{l_2^2 - l_1^2},$$
$$\frac{\partial l_1}{\partial \rho} = \frac{a l_2 - \rho l_1}{l_2^2 - l_1^2} = \frac{\rho(a^2 - l_1^2)}{l_1(l_2^2 - l_1^2)}, \quad \frac{\partial l_2}{\partial \rho} = \frac{\rho l_2 - a l_1}{l_2^2 - l_1^2} = \frac{\rho(l_2^2 - a^2)}{l_2(l_2^2 - l_1^2)}. \tag{A3}$$

Here are some derivatives used in the paper:

$$\frac{\partial}{\partial z}(l_2^2 - a^2)^{1/2} = \frac{l_2(l_2^2 - \rho^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A4}$$

$$\Lambda(l_2^2 - a^2)^{1/2} = \frac{\rho e^{i\phi}(l_2^2 - a^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A5}$$

$$\Lambda\frac{\partial}{\partial z}(l_2^2 - a^2)^{1/2} = \frac{z[a^2(2a^2 + 2z^2 - \rho^2) - l_2^4]}{(l_2^2 - a^2)^{1/2}(l_2^2 - l_1^2)^3}, \tag{A6}$$

$$\frac{\partial^2}{\partial z^2}(l_2^2 - a^2)^{1/2} = \frac{a^2(l_2^2 - \rho^2)^{1/2}}{(l_2^2 - l_1^2)^3}(l_2^2 + 3l_1^2 - 4\rho^2), \tag{A7}$$

$$\frac{\partial}{\partial z}(a^2 - l_1^2)^{1/2} = \frac{l_1(\rho^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A8}$$

$$\Lambda(a^2 - l_1^2)^{1/2} = -\frac{\rho e^{i\phi}(a^2 - l_1^2)^{1/2}}{l_2^2 - l_1^2}, \tag{A9}$$

$$\Lambda\frac{\partial}{\partial z}(a^2 - l_1^2)^{1/2} = -\frac{\rho e^{i\phi} z[l_1^4 - a^2(2a^2 + 2z^2 - \rho^2)]}{(a^2 - l_1^2)^{1/2}(l_2^2 - l_1^2)^3}, \tag{A10}$$

$$\frac{\partial^2}{\partial z^2}(a^2 - l_1^2)^{1/2} = \frac{a^2(\rho^2 - l_1^2)^{1/2}}{(l_2^2 - l_1^2)^3}(4\rho^2 - l_1^2 - 3l_2^2). \tag{A11}$$

$$\frac{\partial}{\partial a}(a^2-l_1^2)^{1/2} = \frac{l_2(l_2^2-\rho^2)^{1/2}}{l_2^2-l_1^2} = \frac{\partial}{\partial z}(l_2^2-a^2)^{1/2}. \tag{A12}$$

$$\frac{\partial}{\partial z}\sin^{-1}\left(\frac{a}{l_2}\right) = -\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}. \tag{A13}$$

$$\Lambda \sin^{-1}\left(\frac{a}{l_2}\right) = -\frac{l_1(l_2^2-a^2)^{1/2}}{l_2[l_2^2-l_1^2]}e^{i\phi}. \tag{A14}$$

$$\Lambda^2 \sin^{-1}\left(\frac{a}{l_2}\right) = \frac{ae^{2i\phi}(l_2^2-a^2)^{1/2}}{l_2^2[l_2^2-l_1^2]^3}[3\rho^2 l_2^2+\rho^2 l_1^2-6a^2\rho^2+2l_1^4]. \tag{A15}$$

$$\frac{\partial^2}{\partial z^2}\sin^{-1}\left(\frac{a}{l_2}\right) = -\frac{\partial}{\partial z}\left(\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right) = \frac{z[a^2(2a^2+2z^2-\rho^2)-l_1^4]}{(a^2-l_1^2)^{1/2}(l_2^2-l_1^2)^3}. \tag{A16}$$

$$\Lambda\frac{\partial}{\partial z}\sin^{-1}\left(\frac{a}{l_2}\right) = -\Lambda\left(\frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\right) = \frac{\rho e^{i\phi}(a^2-l_1^2)^{1/2}}{(l_2^2-l_1^2)^3}[3l_2^2+l_1^2-4a^2]. \tag{A17}$$

Here we present some indefinite integrals of expressions containing l_1 and l_2.

$$\int (l_2^2-a^2)^{1/2}\,dz = (a^2-l_1^2)^{1/2}\frac{l_2^2-2a^2}{2a} + \frac{\rho^2}{2}\ln[l_2+(l_2^2-\rho^2)^{1/2}]. \tag{A18}$$

$$\int (l_2^2-a^2)^{1/2} l_1^2\,dz = -a(a^2-l_1^2)^{1/2}\frac{l_1^2+2a^2}{3} + a^2\rho^2\ln[l_2+(l_2^2-\rho^2)^{1/2}]. \tag{A19}$$

$$\int (a^2-l_1^2)^{1/2}\,dz = \frac{2a^2-l_1^2}{2a}(l_2^2-a^2)^{1/2} + \frac{\rho^2}{2}\sin^{-1}\left(\frac{a}{l_2}\right). \tag{A20}$$

$$\int (a^2-l_1^2)^{1/2} l_1^2\,dz = -\frac{l_1^2(2l_1^2+3\rho^2)}{8a}(l_2^2-a^2)^{1/2} + \rho^2(\tfrac{3}{8}\rho^2-a^2)\sin^{-1}\left(\frac{a}{l_2}\right). \tag{A21}$$

$$\int l_2^2(a^2-l_1^2)^{1/2}\,dz = \tfrac{1}{3}a(l_2^2-a^2)^{1/2}(2a^2+l_2^2) + a^2\rho^2\sin^{-1}\left(\frac{a}{l_2}\right). \tag{A22}$$

$$\int (l_2^2-a^2)^{1/2}\frac{l_1^2}{l_2^2}\,dz = a(a^2-l_1^2)^{1/2}\left[1-\frac{8a^2}{15\rho^2}-\frac{4a^2+3l_1^2}{15l_2^2}\right]. \tag{A23}$$

$$\int \frac{(a^2-l_1^2)^{1/2}}{l_2^2-l_1^2}\,dz = -\sin^{-1}\left(\frac{a}{l_2}\right). \tag{A24}$$

$$\int \frac{(a^2-l_1^2)^{1/2}}{l_2^2(l_2^2-l_1^2)}\,dz = \frac{1}{2a^2}\left[\frac{a(l_2^2-a^2)^{1/2}}{l_2^2}-\sin^{-1}\left(\frac{a}{l_2}\right)\right]. \tag{A25}$$

$$\int \sin^{-1}\left(\frac{a}{l_2}\right)dz = z\sin^{-1}\left(\frac{a}{l_2}\right)-(a^2-l_1^2)^{1/2}+a\ln[l_2+(l_2^2-\rho^2)^{1/2}]. \tag{A26}$$

$$\int z\sin^{-1}\left(\frac{a}{l_2}\right)dz = \tfrac{1}{4}(2a^2+2z^2+\rho^2)\sin^{-1}\left(\frac{a}{l_2}\right)+(l_2^2-a^2)^{1/2}\frac{2a^2+l_1^2}{4a}. \tag{A27}$$

$$\int z^2\sin^{-1}\left(\frac{a}{l_2}\right)dz = \tfrac{1}{3}z^3\sin^{-1}\left(\frac{a}{l_2}\right)+\tfrac{1}{18}(a^2-l_1^2)^{1/2}(3l_2^2+6\rho^2+8a^2-2l_1^2)-\tfrac{1}{6}a(3\rho^2+2a^2)\ln[l_2+(l_2^2-\rho^2)^{1/2}]. \tag{A28}$$

The integration in (A18)–(A28) was performed by parts, with a consequent change of variables: $z = (a^2-l_1^2)^{1/2}(\rho^2-l_1^2)^{1/2}/l_1$ or $z = (l_2^2-a^2)^{1/2}(l_2^2-\rho^2)^{1/2}/l_2$.